NEW METHOD OF SOLVING PROBLEMS OF RADIATION TRANSPORT THEORY.

1. GENERAL SOLUTION. SEMIINFINITE LAYER

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An approximate solution is proposed for the radiation transport equations in a dissipating medium which is that the radiation intensity experiencing two and more scattering acts is found in the transport approximation. The approximation developed possesses high accuracy and permits obtaining analytic expressions in those cases where numerical computations had been used earlier.

As is known [1-3], investigations of radiation transport in two-phase media are of great theoretical and practical value for many divisions of physics and its engineering applications. The solution of the transport equation in the general case is associated with great mathematical difficulties, especially when it is required to take account of anisotropy of scattering, the reflective properties of the boundary surfaces, the geometry of the medium under investigation, etc. In this paper, the solution of the radiation transport equation is expressed in terms of the intensity of single scattering and the intensity of second and higher multiplicity scattering. The correction to the multiplicity of scattering is described by a function determined by the method developed in [4-6]. Problems of radiation propagation in semiinfinite and finite media with isotropic scattering, a semiinfinite layer with anisotropic scattering, a planar medium with known reflection properties of the boundary surfaces, and also the case of conservative scattering will be examined below. A comparison with the results of numerical computations and analytic solutions will indicate the high accuracy of the method proposed.

Let us consider the propagation of radiation in a homogeneous plane layer with a given distribution of internal radiation sources for an arbitrary scattering index:

$$\mu \frac{dJ(\tau, \mu)}{d\tau} + J(\tau, \mu) = \frac{\lambda}{2} \int_{-1}^{1} p(\mu, \mu') J(\tau, \mu')\, d\mu' + A(\tau) + \lambda B(\tau). \tag{1}$$

Here $d\tau = (\varkappa + \sigma)dx$ is the elementary optical thickness of the layer, and the function $A(\tau) + \lambda B(\tau)$ characterizes the internal radiation source function in general form. Thus, upon compliance with the local thermodynamic equilibrium condition

$$A(\tau) + \lambda B(\tau) = (1 - \lambda)\, B_\nu [T(\tau)], \tag{2}$$

where $B_\nu[T(\tau)]$ is the intensity of Planck radiation of frequency ν at a temperature $T = T(\tau)$. The boundary conditions for (1) take account of the presence of external radiation passing through the boundary surfaces and the radiation they reflect that emerges from the medium

$$\begin{gathered} J(0, +\mu) = J_{01}(+\mu) + \frac{1}{\mu} \int_0^1 y_1(\mu, \mu') J(0, -\mu') \mu' d\mu', \\ J(\tau_0, -\mu) = J_{02}(-\mu) + \frac{1}{\mu} \int_0^1 y_2(\mu, \mu') J(\tau_0, +\mu') \mu' d\mu'. \end{gathered} \tag{3}$$

The reflectivities of the boundary surfaces $y_i(\mu, \mu')$ $(i = 1, 2)$ are analogously introduced [1]. In the diffuse reflection case, these quantities equal

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$$y_i(\mu, \mu') = 2A_i\mu, \quad (4)$$

while for Fresnel (or specular) reflection

$$y_i(\mu, \mu') = r_i(\mu')\,\delta(\mu - \mu'). \quad (5)$$

To solve (1) under the boundary conditions (3) we turn to successive approximations, i.e., we seek the solution in the form [1, 7]

$$J(\tau, \mu) = \sum_{n=0}^{\infty} \lambda^n J^{(n)}(\tau, \mu), \quad (6)$$

where $J^{(n)}(\tau, \mu)$ is the radiation intensity scattered n times. Substituting (6) into (1), the equation and boundary conditions to determine $J^{(n)}(\tau, \mu)$ are easily written:

$$\mu \frac{dJ^{(0)}(\tau, \mu)}{d\tau} + J^{(0)}(\tau, \mu) = A(\tau),$$

$$J^{(0)}(0, +\mu) = J_{01}(+\mu) + \frac{1}{\mu}\int_0^1 y_1(\mu, \mu')\,J^{(0)}(0, -\mu')\,\mu' d\mu', \quad (7)$$

$$J^{(0)}(\tau_0, -\mu) = J_{02}(-\mu) + \frac{1}{\mu}\int_0^1 y_2(\mu, \mu')\,J^{(0)}(\tau_0, +\mu')\,\mu' d\mu',$$

$$\mu \frac{dJ^{(1)}(\tau, \mu)}{d\tau} + J^{(1)}(\tau, \mu) = \frac{1}{2}\int_{-1}^1 p(\mu, \mu')\,J^{(0)}(\tau, \mu')\,d\mu' + B(\tau), \quad (8)$$

$$\mu \frac{dJ^{(m)}(\tau, \mu)}{d\tau} + J^{(m)}(\tau, \mu) = \frac{1}{2}\int_{-1}^1 p(\mu, \mu')\,J^{(m-1)}(\tau, \mu')\,d\mu' \; (m = 2, 3, \ldots). \quad (9)$$

The boundary conditions for (8) and (9) are identical in form

$$J^{(m)}(0, +\mu) = \frac{1}{\mu}\int_0^1 y_1(\mu, \mu')\,J^{(m)}(0, -\mu')\,\mu' d\mu',$$

$$J^{(m)}(\tau_0, -\mu) = \frac{1}{\mu}\int_0^1 y_2(\mu, \mu')\,J^{(m)}(\tau_0, +\mu')\,\mu' d\mu'. \quad (10)$$

According to (7)-(10), to determine the functions

$$f(\tau, \mu) = J(\tau, \mu) - J^{(0)}(\tau, \mu) - \lambda J^{(1)}(\tau, \mu) = \sum_{m=2}^{\infty} \lambda^m J^{(m)}(\tau, \mu) \quad (11)$$

we obtain the following equation:

$$\mu \frac{df(\tau, \mu)}{d\tau} + f(\tau, \mu) = \frac{\lambda}{2}\int_{-1}^1 p(\mu, \mu')\,f(\tau, \mu')\,d\mu' + G(\tau, \mu), \quad (12)$$

where

$$G(\tau, \mu) = \frac{\lambda^2}{2}\int_{-1}^1 p(\mu, \mu')\,J^{(1)}(\tau, \mu')\,d\mu'. \quad (13)$$

The boundary conditions for this equation are analogous to (10).

The function $f(\tau, \mu)$ introduced characterizes the contribution of second and higher multiplicity scattering. According to (12) its magnitude is determined by $G(\tau, \mu)$, i.e., the intensity of singly scattered radiation. This function permits direct estimation of the role of multiple scattering processes:

TABLE 1. Comparison of the Exact (φ) [1] and Computed Values (φ_C) from (9) for the Ambartsumyan Function $\varphi(\mu)$

μ	λ = 0,4		0,5		0,6		0,7		0,8		0,9		0,95	
	φ	φ_C	φ	φ_C	φ	φ_C	φ	φ_C	φ	φ_C	φ	φ_C	φ	φ_C
0	1,00	1,00	1,00	1,00	1,00	1,00	1,00	1,00	1,00	1,00	1,00	1,00	1,00	1,00
0,1	1,06	1,05	1,07	1,07	1,09	1,09	1,11	1,11	1,14	1,13	1,17	1,16	1,19	1,18
0,2	1,09	1,08	1,11	1,11	1,15	1,14	1,18	1,17	1,23	1,22	1,29	1,28	1,34	1,32
0,3	1,11	1,11	1,14	1,14	1,19	1,18	1,24	1,23	1,30	1,29	1,39	1,38	1,46	1,44
0,4	1,13	1,12	1,17	1,16	1,22	1,21	1,28	1,27	1,36	1,35	1,48	1,46	1,57	1,55
0,5	1,14	1,14	1,19	1,18	1,25	1,24	1,32	1,31	1,41	1,40	1,56	1,54	1,67	1,66
0,6	1,15	1,15	1,20	1,20	1,27	1,26	1,35	1,34	1,46	1,45	1,63	1,61	1,76	1,75
0,7	1,16	1,16	1,22	1,22	1,29	1,28	1,38	1,37	1,50	1,49	1,69	1,68	1,85	1,84
0,8	1,17	1,17	1,23	1,23	1,31	1,30	1,40	1,40	1,54	1,53	1,75	1,74	1,93	1,93
0,9	1,18	1,18	1,24	1,24	1,32	1,32	1,42	1,42	1,57	1,56	1,80	1,80	2,01	2,01
1,0	1,18	1,18	1,25	1,25	1,34	1,33	1,44	1,44	1,60	1,59	1,85	1,85	2,08	2,08

$$\Delta = \Delta(\tau, \mu) = \frac{f(\tau, \mu)}{\lambda J^{(1)}(\tau, \mu)}, \tag{14}$$

and the solution of the initial equation (1) is itself written in terms of the magnitude of the single scattered radiation intensity:

$$J(\tau, \mu) = J^{(0)}(\tau, \mu) + \lambda(1 + \Delta) J^{(1)}(\tau, \mu). \tag{15}$$

We call the function $f(\tau, \mu)$ the multiple scattering function.

Photons experiencing two and more acts of scattering produce a more-or-less uniform angular distribution of the intensity in the medium. In this case, as is shown in [4-7], the transport approximation can be used, i.e., the scattering index is represented in the form

$$p(\mu, \mu') = a + 2(1 - a)\,\delta(\mu - \mu'). \tag{16}$$

Then (12) takes the form

$$\mu \frac{df(t, \mu)}{dt} + f(t, \mu) = \frac{l}{2} \int_{-1}^{1} f(t, \mu')\, d\mu' + \beta G(t, \mu). \tag{17}$$

Here

$$\beta = \frac{1}{1 - \lambda(1 - a)}; \quad l = \frac{a\lambda}{1 - \lambda(1 - a)}; \quad dt = \frac{1}{\beta}\, d\tau. \tag{18}$$

In the case $a = 1$, i.e., when the index is spherical, $\beta = 1$, $l = \lambda$, $t = \tau$. To solve (17), we use the method developed in [4-6]. Let us introduce the function

$$f_1(t) = \int_0^1 f(t, +\mu)\, d\mu, \quad f_2(t) = \int_0^1 f(t, -\mu)\, d\mu. \tag{19}$$

In the Schwarzschild–Schuster approximation we have the following system of equations for these functions

$$\begin{aligned} \frac{1}{2} \frac{df_1(t)}{dt} + f_1(t) &= \frac{l}{2} [f_1(t) + f_2(t)] + g_1(t), \\ -\frac{1}{2} \frac{df_2(t)}{dt} + f_2(t) &= \frac{l}{2} [f_1(t) + f_2(t)] + g_2(t), \end{aligned} \tag{20}$$

where

$$g_1(t) = \beta \int_0^1 G(t, +\mu)\, d\mu, \quad g_2(t) = \beta \int_0^1 G(t, -\mu)\, d\mu. \tag{21}$$

Using the notation

$$I(t) = f_1(t) + f_2(t), \quad H(t) = f_1(t) - f_2(t), \tag{22}$$

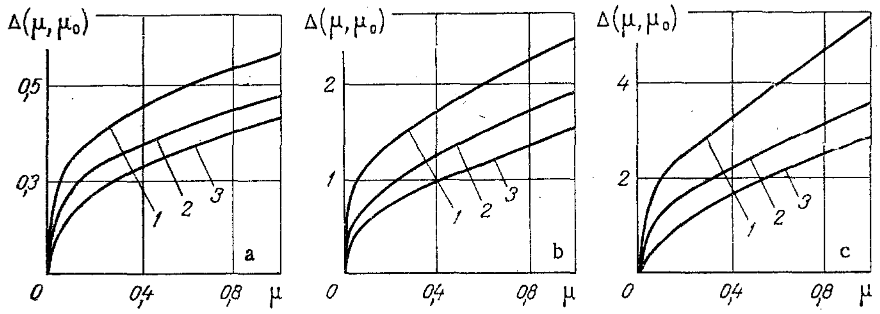

Fig. 1. Dependence of the correction to multiple scattering on the angle of observation: a) $\lambda = 0.5$; b) $\lambda = 0.9$; c) $\lambda = 0.99$; 1) $\mu_0 = 1.0$; 2) 0.5; 3) 0.3.

we find

$$\frac{d^2 I(t)}{dt^2} - k^2 I(t) = -2h(t), \quad H(t) = -\frac{1}{2}\frac{dI(t)}{dt} + g_1(t) - g_2(t). \tag{23}$$

Here

$$k^2 = 4(1-l), \quad h(t) = 2[g_1(t) + g_2(t)] - \frac{d}{dt}[g_1(t) - g_2(t)]. \tag{24}$$

The solutions of (23) are

$$I(t) = A_1 e^{-k(t_0 - t)} + A_2 e^{-kt} + \Phi_1(t) + \Phi_2(t),$$

$$H(t) = \frac{k}{2}[-A_1 e^{-k(t_0-t)} + A_2 e^{-kt} - \Phi_1(t) + \Phi_2(t)] + g_1(t) - g_2(t),$$

where

$$\Phi_1(t) = \frac{1}{k}\int_t^{t_0} h(t')\, e^{-k(t'-t)}\, dt'; \quad \Phi_2(t) = \frac{1}{k}\int_0^{t} h(t')\, e^{-k(t-t')}\, dt'. \tag{25}$$

According to (22), expressions for $f_1(t)$ and $f_2(t)$ follow

$$f_1(t) = \alpha_1 A_1 e^{-k(t_0-t)} + \alpha_2 A_2 e^{-kt} + \alpha_1 \Phi_1(t) + \alpha_2 \Phi_2(t) + \frac{1}{2}[g_1(t) - g_2(t)],$$
$$f_2(t) = \alpha_2 A_1 e^{-k(t_0-t)} + \alpha_1 A_2 e^{-kt} + \alpha_2 \Phi_1(t) + \alpha_1 \Phi_2(t) - \frac{1}{2}[g_1(t) - g_2(t)], \tag{26}$$

where

$$\alpha_1 = \frac{1}{4}(2-k), \quad \alpha_2 = \frac{1}{4}(2+k). \tag{27}$$

Integrating the relationship (10) for $f(t, \mu)$ with respect to μ, we find the boundary conditions for the functions $f_1(t)$ and $f_2(t)$:

$$f_1(0) = \rho_1 f_2(0) \text{ and } f_2(t_0) = \rho_2 f_1(t_0). \tag{28}$$

Some effective reflectivities of the boundary surfaces are introduced here:

$$\rho_1 = \frac{\int_0^1 \frac{d\mu}{\mu}\int_0^1 y_1(\mu, \mu')\, f(0, -\mu')\, \mu' d\mu'}{\int_0^1 f(0, -\mu')\, d\mu},$$

$$\rho_2 = \frac{\int_0^1 \frac{d\mu}{\mu} \int_0^1 y_2(\mu, \mu') f(t_0, +\mu') \mu' d\mu'}{\int_0^1 f(t_0, +\mu)\, d\mu}. \tag{29}$$

Conditions (28) permit determination of the constants A_1 and A_2

$$A_1 = -\frac{M_2(1-\rho_1 R) - M_1(R-\rho_2)e^{-kt_0}}{(1-\rho_1 R)(1-\rho_2 R) - (R-\rho_1)(R-\rho_2)e^{-2kt_0}},$$
$$A_2 = -\frac{M_1(1-\rho_2 R) - M_2(R-\rho_1)e^{-kt_0}}{(1-\rho_1 R)(1-\rho_2 R) - (R-\rho_1)(R-\rho_2)e^{-2kt_0}}, \tag{30}$$

where

$$M_1 = (R-\rho_1)\Phi_1 + \frac{1}{2}(1+R)(1+\rho_1)[g_1(0) - g_2(0)];$$
$$M_2 = (R-\rho_2)\Phi_2 - \frac{1}{2}(1+R)(1+\rho_2)[g_1(t_0) - g_2(t_0)]; \tag{31}$$

$$\Phi_1 = \Phi_1(0) = \frac{1}{k}\int_0^{t_0} h(t')e^{-kt'}dt';\quad \Phi_2 = \Phi_2(t_0) = \frac{1}{k}\int_0^{t_0} h(t')e^{-k(t_0-t')}dt';$$

$$R = \frac{\alpha_1}{\alpha_2} = \frac{2-k}{2+k} = \frac{1-\sqrt{1-l}}{1+\sqrt{1-l}}.$$

Let us note that the quantity R is the reflectivity of a semiinfinite scattering layer [4].

The expressions (26) are of definite interest for investigation since they are multiple scattering functions averaged over the positive and negative hemispheres. Nevertheless, we turn to determining the function $f(t, \mu)$ itself. Substituting the expression obtained for $I(t)$ into (17) instead of the integral, we find

$$\mu \frac{df(t, \mu)}{dt} + f(t, \mu) = \frac{l}{2}[A_1 e^{-k(t_0-t)} + A_2 e^{-kt} + \Phi_1(t) + \Phi_2(t)] + \beta G(t, \mu). \tag{32}$$

Hence, the desired solution is determined by the following expressions

$$f(t, +\mu) = f(0, +\mu)e^{-\frac{t}{\mu}} + F_1(t, \mu),\ f(t, -\mu) = f(t_0, -\mu)e^{-\frac{t_0-t}{\mu}} + F_2(t, \mu), \tag{33}$$

where

$$F_1(t, \mu) = \frac{lA_1}{2}\cdot\frac{e^{-k(t_0-t)} - e^{-kt_0-\frac{t}{\mu}}}{1+k\mu} + \frac{lA_2}{2}\cdot\frac{e^{-kt} - e^{-\frac{t}{\mu}}}{1-k\mu}$$
$$+\frac{l\Phi_2(t)}{2(1-k\mu)} + \frac{l}{2(1+k\mu)}[\Phi_1(t) - \Phi_1 e^{-\frac{t}{\mu}}] - \frac{lk^2\mu^2}{1-k^2\mu^2}\mathcal{P}_1(t, \mu) + Q_1(t, \mu); \tag{34}$$

$$F_2(t, \mu) = \frac{lA_1}{2}\cdot\frac{e^{-k(t_0-t)} - e^{-\frac{t_0-t}{\mu}}}{1-k\mu} + \frac{lA_2}{2}\,\frac{e^{-kt} - e^{-kt_0-\frac{t_0-t}{\mu}}}{1+k\mu} + \frac{l\Phi_1(t)}{2(1-k\mu)} \tag{35}$$
$$+\frac{l}{2(1+k\mu)}[\Phi_2(t) - \Phi_2 e^{-\frac{t_0-t}{\mu}}] - \frac{lk^2\mu^2}{1-k^2\mu^2}\mathcal{P}_2(t, \mu) + Q_2(t, \mu);$$

$$\mathcal{P}_1(t, \mu) = \frac{1}{k^2}\int_0^t h(t')e^{-\frac{t-t'}{\mu}}\frac{dt'}{\mu};$$
$$\mathcal{P}_2(t, \mu) = \frac{1}{k^2}\int_t^{t_0} h(t')e^{-\frac{t'-t}{\mu}}\frac{dt'}{\mu}; \tag{36}$$

$$Q_1(t, \mu) = \beta \int_0^t G(t', \mu) e^{-\frac{t-t'}{\mu}} \frac{dt'}{\mu};$$

$$Q_2(t, \mu) = \beta \int_t^{t_0} G(t', -\mu) e^{-\frac{t'-t}{\mu}} \frac{dt'}{\mu}. \tag{37}$$

The quantities $f(0, +\mu)$ and $f(t_0, -\mu)$ in (33) are easily determined when definite laws of boundary surface reflection are given. Thus, in the diffuse reflection case, we find according to (4)

$$f(0, +\mu) = 2A_1 \int_0^1 f(0, -\mu') \mu' d\mu' \cong A_1 f_2(0),$$

$$f(t_0, -\mu) = 2A_2 \int_0^1 f(t_0, +\mu') \mu' d\mu' \cong A_2 f_1(t_0), \tag{38}$$

where $f_1(t_0)$ and $f_2(0)$ are determined by (26). For Fresnel (or specular) reflection we obtain the exact expressions

$$f(0, +\mu) = \frac{r_1(\mu)}{1 - r_1(\mu) r_2(\mu) e^{-\frac{2t_0}{\mu}}} \left[F_2(0, \mu) + r_2(\mu) e^{-\frac{t_0}{\mu}} F_1(t_0, \mu)\right],$$

$$f(t_0, -\mu) = \frac{r_2(\mu)}{1 - r_1(\mu) r_2(\mu) e^{-\frac{2t_0}{\mu}}} \left[F_1(t_0, \mu) + r_1(\mu) e^{-\frac{t_0}{\mu}} F_2(0, \mu)\right]. \tag{39}$$

If the boundary surfaces reflect the radiation by different laws, then

$$f(0, +\mu) = A_1 f_2(0), \; f(t_0, -\mu) = r_2(\mu) \left[F_1(t_0, \mu) + A_1 f_2(0) e^{-\frac{t_0}{\mu}}\right] \tag{40a}$$

or

$$f(0, +\mu) = r_1(\mu) \left[F_2(0, \mu) + A_2 f_1(t_0) e^{-\frac{t_0}{\mu}}\right], \; f(t_0, -\mu) = A_2 f_1(t_0). \tag{40b}$$

Naturally, if the boundary surfaces reflect no radiation (or almost none) the quantities under consideration vanish.

Let us turn to the problem of diffuse reflection of radiation from a semiinfinite medium under isotropic scattering. In this case, according to (33) the multiple scattering function takes the form

$$f_0 = f(0, -\mu)|_{t_0 \to \infty} = \frac{lA_2}{2(1 + k\mu)} + \frac{l}{2k(1 - k\mu)} \int_0^\infty h(t') e^{-kt'} dt'$$
$$- \frac{l\mu^2}{1 - k^2\mu^2} \int_0^\infty h(t') e^{-\frac{t'}{\mu}} \frac{dt'}{\mu} + \beta \int_0^\infty G(t', -\mu) e^{-\frac{t'}{\mu}} \frac{dt'}{\mu}$$

or

$$f_0 = \beta \int_0^\infty G(t', -\mu) e^{-\frac{t'}{\mu}} \frac{dt'}{\mu} + \frac{l}{2k} \left(\frac{1}{1 - k\mu} - \frac{R}{1 + k\mu}\right)$$
$$\times \int_0^\infty h(t') e^{-kt'} dt' - \frac{l\mu^2}{1 - k^2\mu^2} \int_0^\infty h(t') e^{-\frac{t'}{\mu}} \frac{dt'}{\mu}. \tag{41}$$

For specific computations of the integrals in the last expression, the quantity $J^{(1)}(t, \mu)$ must be known, and the condition for isotropy of the scattering must be used. The problem of determining $J^{(1)}(t, \mu)$ for a semiinfinite layer is quite simple. According to (7) and (8), it is easy to obtain that upon incidence of external radiation of intensity J_0 at an angle $\theta_0 = \arccos\mu_0$ to the normal of the layer

$$J^{(0)}(\tau, +\mu) = J_0\delta(\mu - \mu_0)\, e^{-\frac{\tau}{\mu}},\quad J^{(0)}(\tau, -\mu) = 0,$$

$$J^{(1)}(\tau, +\mu) = \frac{\mu_0 J_0 p(\mu, \mu_0)}{4(\mu_0 - \mu)} \left(e^{-\frac{\tau}{\mu_0}} - e^{-\frac{\tau}{\mu}}\right),$$

$$J^{(1)}(\tau, -\mu) = \frac{\mu_0 J_0 p(-\mu, \mu_0)}{4(\mu_0 + \mu)} \left(e^{-\frac{\tau}{\mu_0}} - e^{-\frac{\tau_0}{\mu_0} - \frac{\tau_0 - \tau}{\mu}}\right),$$

$$J^{(1)}(\tau, -\mu)\Big|_{\tau_0 \to \infty} = \frac{\mu_0 J_0 p(-\mu, \mu_0)}{4(\mu_0 + \mu)}\, e^{-\frac{\tau}{\mu_0}}. \tag{42}$$

Using the condition of scattering isotropy, we find

$$G(t, \mu) = \frac{\lambda^2}{2} \int_{-1}^{1} J^{(1)}(\tau, \mu)\, d\mu = G(\tau),\quad g_1(t) = g_2(t) = G(\tau),$$

$$h(t) = 4G(\tau).$$

Then

$$f_0 = \frac{1 - 4\mu^2}{1 - k^2\mu^2}\, I\left(\frac{1}{\mu}\right) + \frac{2\lambda}{k^2}\left(\frac{1}{1 - k\mu} - \frac{R}{1 + k\mu}\right) I_0(k), \tag{43}$$

where

$$I_0(k) = k \int_0^\infty G(\tau')\, e^{-k\tau'}\, d\tau'. \tag{44}$$

Substituting the explicit expression for the function $G(\tau)$ in (44), we obtain

$$I_0(k) = \frac{\lambda^2}{8} \cdot \frac{k\mu_0 J_0}{1 + k\mu_0}\left[\mu_0 \ln \frac{1 + \mu_0}{\mu_0} + \frac{1}{k} \ln(1 + k)\right]. \tag{44a}$$

Therefore, the solution of the problem of diffuse reflection of radiation from a semi-infinite medium can be written in the form

$$J(0, -\mu) = \lambda(1 + \Delta)\, J^{(1)}(0, -\mu), \tag{45}$$

where

$$\Delta = \Delta(\mu, \mu_0) = \frac{\lambda}{2}\, \frac{1 - 4\mu^2}{1 - k^2\mu^2}\left(\mu_0 \ln \frac{1 + \mu_0}{\mu_0} + \mu \ln \frac{1 + \mu}{\mu}\right) +$$

$$+ \frac{\lambda^2(\mu_0 + \mu)}{k(1 + k\mu_0)}\left(\frac{1}{1 - k\mu} - \frac{R}{1 + k\mu}\right)\left[\mu_0 \ln \frac{1 + \mu_0}{\mu_0} + \frac{1}{k} \ln(1 + k)\right]. \tag{46}$$

The coefficient of diffuse reflection for a semiinfinite medium is determined by the expression:

$$\rho(\mu, \mu_0) = \frac{J(0, -\mu)}{J_0\mu_0} = \frac{\lambda(1 + \Delta)}{4(\mu_0 + \mu)}. \tag{47}$$

We have obtained the solution of the problem in the form (45) or (47) in terms of the single scattering approximation. The function f_0 (or Δ) permits a numerical estimation of the contribution of multiple scattering to the total intensity of scattered radiation as a function of the optical properties of the medium and the conditions of performing the experiment. Let us note that the limits of applicability of the single scattering approximation are determined by the quantity Δ.

The error in the method proposed for this problem is established sufficiently simply since the exact solution is already known [1]

$$\rho(\mu, \mu_0) = \frac{\lambda}{4} \cdot \frac{\varphi(\mu_0)\varphi(\mu)}{\mu_0 + \mu}, \tag{48}$$

where $\varphi(\mu)$ is the Ambartsumyan function defined by the integral expression

$$\varphi(\mu)=1+\frac{\lambda}{2}\mu\varphi(\mu)\int_0^1\frac{\varphi(\mu')}{\mu+\mu'}d\mu'.$$

On the other hand, setting $\mu=\mu_0$ in (47) and (48), we have

$$\varphi_c^2(\mu)=1+\Delta(\mu,\mu)\ \text{or}\ \varphi_c(\mu)=\sqrt{1+\Delta_0(\mu)}, \tag{49}$$

$$\Delta_0(\mu)=\Delta(\mu,\mu)=\lambda\mu\frac{1-4\mu^2}{1-k^2\mu^2}\ln\frac{1+\mu}{\mu}+\frac{2\lambda^2\mu}{k(1+k\mu)}\left(\frac{1}{1-k\mu}-\frac{R}{1+k\mu}\right)\left[\mu\ln\frac{1+\mu}{\mu}+\frac{1}{k}\ln(1+k)\right]. \tag{50}$$

The data for computing $\varphi_n(\mu)$ and the exact values of $\varphi(\mu)$ are presented in the table for different values of the probability of survival of a quantum λ. Analysis of these data shows that the error in the method proposed is on the order of 1% in the determination of the Ambartsumyan function and therefore in the solution of the transport equation.

The dependence of the correction for multiple scattering $\Delta(\mu, \mu_0)$ on the angle of observation is represented in the figure for different values of the survival probability of a quantum λ and angle of external radiation incidence θ_0. Analysis of the dependences presented shows that for strongly scattering media ($\lambda \geqslant 0.9$) in a broad range of variation of the angle of observation ($\mu \cong 0.2$-1.0), the quantity $\Delta(\mu, \mu_0)$ depends linearly on μ. The angular coefficient of the lines $\Delta = c\mu$ grows with the growth of both λ and μ_0. Let us note that according to (15) the domain of applicability of the single scattering approximation is determined by the value

$$E=\frac{J(0,-\mu)-\lambda J^{(1)}(0,-\mu)}{J(0,-\mu)}=\frac{\Delta}{1+\Delta}. \tag{51}$$

Thus, for $\mu=\mu_0=1$ the quantity E equals

λ	0.4	0,5	0,6	0,7	0,8	0.9	0,95	0,975	0,99	0,995
$E\cdot 100$ %	28,4	35,9	43.7	51,9	60,6	70,7	77.0	80,9	84,1	85,6

For strongly dissipating media ($\lambda = 1-\delta^2$, $\delta \ll 1$), its approximate representation

$$\Delta_0(\mu)\cong\mu(1-4\mu^2)\ln\frac{1+\mu}{\mu}+\frac{2(1+\mu\delta)}{(1+\delta)(1+2\mu\delta)}\left(1+\mu\ln\frac{1+\mu}{\mu}\right) \tag{52}$$

can be used in place of (50).

In conclusion, let us note that the case when $\mu\to 1/k$ must be examined specially. As is easy to show

$$f_0\Big|_{\mu\to\frac{1}{k}}=\frac{\lambda^2}{16k^2}(k+2)J_0\ln(1+k). \tag{53}$$

It is later planned to analyze the accuracy of the proposed method in examining a finite layer and taking account of the effect of anisotropy, as well as to show the possibility of taking account of the reflective properties of the boundary surfaces for important practical cases.

NOTATION

$J(\tau, \mu)$, radiation intensity at the point τ and the direction $\theta = \arccos \mu$; $p(\mu, \mu')$, scattering index in a volume element; $\varkappa$, absorption coefficient; σ, scattering coefficient; $\lambda = \sigma/(\varkappa+\sigma)$, probability of survival of a quantum; τ, optical depth; $B_\nu(T)$, Planck function; T, temperature; $y_i(\mu, \mu')$, reflectivity of the boundary i; $f(\tau, \mu)$, radiation intensity scattered two and more times; a, twice the hemispherical backscattered fraction.

LITERATURE CITED

1. V. V. Sobolev, Radiant Energy Transport in the Atmospheres of Stars and Planets [in Russian], GITTL, Moscow (1956).
2. V. N. Adrianov, Principles of Radiational and Complex Heat Transfer [in Russian], Énergiya, Moscow (1972).

3. M. N. Ozisik, Basic Heat Transfer, McGraw-Hill (1977).
4. K. S. Adzerikho and V. P. Nekrasov, "Computation of the glow characteristics of light-scattering media," Inzh.-Fiz. Zh., 22, No. 1, 168-170 (1972).
5. K. S. Adzerikho and V. P. Nekrasov, "Luminescence characteristics of cylindrical and spherical light scattering media," Int. J. Heat Mass Transfer, 18, No. 9, 1131-1138 (1975).
6. K. S. Adzerikho, V. I. Antsulevich, V. P. Nekrasov, and V. P. Trofimov, "Modeling of radiant heat exchange problems in media of nonplanar geometry," Inzh.-Fiz. Zh., 36, No. 2, 231-243 (1979).
7. K. S. Adzerikho, Lectures on Radiant Energy Transport Theory [in Russian], Belorussian State Univ., Minsk (1975).

ALGORITHM OF THE ZONAL SOLUTION OF RADIATION—CONDUCTION HEAT-TRANSFER PROBLEMS

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A numerical method is proposed to compute the stationary radiation—conduction heat transfer in semitransparent materials on the basis of a zonal approach.

The development of methods to compute the radiation—conduction heat transfer [1] is of great value for many thermal-engineering applications. The use of high-speed electronic computers with sufficient mathematical support permits the execution of a penetrating computational theoretical analysis of this kind of heat transfer in absorbing inhomogeneous media with a detailed accounting of the frequency—temperature dependence of the optical characteristics in both the bulk and on the boundaries of the radiating system [2-5]. Great attention is paid to overcoming the mathematical difficulties in solving radiation—conduction heat-transfer (RCT) problems in the presence of semiopacity of the boundary surfaces [4, 5], as well as moving phase interfaces [6].

It should be noted, however, that the high level of detail achieved in computations in [2-6] is as yet realized for the one-dimensional plane-parallel case. Nevertheless, the need to produce computational methods permitting the analysis of RCT in two- and three-dimensional systems of different configuration is already overdue. Hence, by taking into account the difficulties of realizing exact formulations of complex heat-transfer problems for arbitrary volume geometry conditions, the prospects of approximate zonal methods [1] based on the approximation of the initial radiation integral equations by a system of algebraic equations [7] are noted. Meanwhile, the inadequately extensive utilization of these methods in the theory of complex heat transfer is indicated in [1]. An analysis of foreign investigations of the application of approximate methods of solving complex heat-transfer problems in bulk systems is presented in [8], and reduces to recommendations to utilize the so-called method of generalized angular coefficients in the RCT domain in [8]. The expediency of using the statistical testing (Monte Carlo) method, whose efficiency is demonstrated in a number of examples, is indicated in [8] for the determination of the generalized angular coefficients as well as the radiation exchange coefficients for the solution of different complex heat-transfer problems. In particular, the simplicity and physical nature of the solution of problems with complex bulk geometries of the radiating systems by the method mentioned are noted. The prospects of utilizing the Monte Carlo method to model radiation transport processes are also noted in [1].

The results of trying out the algorithm for the approximate solution of RCT problems on the basis of a zonal approach [7, 9] and the utilization of the Monte Carlo method to determine the radiation exchange coefficients [10, 11], as well as a finite-difference scheme to take account of heat transfer by heat conduction [12] are presented in this paper. The

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